Problem Set 1

This first problem set is designed to help you gain a familiarity with set theory and basic proof techniques.

Start this problem set early. It contains ten problems (one checkpoint question, eight graded problems, plus one survey question), several of which require a fair amount of thought. I would suggest reading through this problem set at least once as soon as you get it to get a sense of what it covers.

As much as you possibly can, please try to work on this problem set individually. If you work too much in a group, you'll miss the chance to strengthen your mathematical muscles.* That said, if you do work with others, please be sure to cite who you are working with and on what problems. For more details, see the section on the Stanford Honor Code in the course information handout.

In any question that asks for a proof, you **must** provide a rigorous mathematical proof. You cannot draw a picture or argue by intuition. You should, at the very least, state what type of proof you are using, and (if proceeding by contradiction or contrapositive) state exactly what it is that you are trying to show. If we specify that a proof must be done a certain way, you must use that particular proof technique; otherwise you may prove the result however you wish.

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

This problem set has 150 possible points. It is weighted at 6% of your total grade. The earlier questions serve as a warm-up for the later problems, so the difficulty of the problems increases over the course of this problem set.

Good luck, and have fun!

Checkpoint Questions Due Monday, April 9 at 2:15 PM
Remaining Questions Due Friday, April 13 at 2:15 PM

* Trust me, they exist. ☺

Checkpoint Question: Multiples of Three (25 Points if Submitted)

Write your solutions to the following problems and submit them by Monday, April 9th at the start of class. These problems will be graded based on whether or not you submit it, rather than the correctness of your solutions. We will try to get these problems returned to you with feedback on your proof style this Wednesday, April 11th. Submission instructions are on the last page of this problem set.

Please make the best effort you can when solving these problems. We want the feedback we give you on your solutions to be as useful as possible, so the more time and effort you put into them, the better we'll be able to comment on your proof style and technique.

A number is a *multiple of three* iff it can be written as $3k$ for some integer k. A number is *congruent to one modulo three* iff it can be written as $3k + 1$ for some integer k, and a number is *congruent to two modulo three* iff it can be written as $3k + 2$ for some integer k. For each integer n, exactly one of the following is true (you don't need to prove this):

- n is a multiple of three.
- n is congruent to one modulo three.
- n is congruent to two modulo three.

Suppose that we want to prove this result:

n is a multiple of three iff n^2 is a multiple of three.

To do this, we will prove the following two statements:

If n is a multiple of three, n^2 is a multiple of three.

If n^2 is a multiple of three, n is a multiple of three.

i. Prove the first of these statements with a direct proof.

ii. Prove the second of these statements using the contrapositive. Make sure that you state the contrapositive of the statement explicitly before you attempt to prove it.

iii. Prove, by contradiction, that $\sqrt{3}$ is irrational. Make sure that you explicitly state what assumption you are making before you derive a contradiction from it. Recall from lecture that a rational number is one that can be written as p / q for integers p and q where $q \neq 0$ and p and q have no common divisor other than ± 1.

The remainder of these problems should be completed and returned by Friday, April 13 at the start of class.

Problem One: Elementary Set Theory (4 points)

For the purposes of this problem, suppose that we are dealing with the following sets:

$A = \{ 1, 2, 3, 4 \}$

$B = \{ 2, 2, 2, 1, 4, 3 \}$

$C = \{ 1, 3 \}$

$D = \{ 2, 3, 4 \}$

$E = \{ x \mid x \in \mathbb{N} \text{ and } x \text{ is even} \}$

For each of the following, is the claim true or false? Explain why. You do not need to prove your assertions.

i. $A = B$.

ii. $C \Delta D = C$

iii. $|D| > |A|$

iv. $E \cap D = C \cap D$

v. $C \in A$.

vi. $C \subseteq A$.

Problem Two: Two Is Irrational? (12 points)

In lecture, we proved that $\sqrt{2}$ is irrational, and in the checkpoint problem you proved that $\sqrt{3}$ is irrational. Below is a purported proof that $\sqrt{4}$ is irrational:

Theorem: $\sqrt{4}$ is irrational.

Proof: By contradiction; assume that $\sqrt{4}$ is rational. Then there must exist integers p, q such that $q \neq 0$, $p / q = \sqrt{4}$, and p and q have no common factors other than 1 and -1.

Since $p / q = \sqrt{4}$, $p^2 / q^2 = 4$, so $p^2 = 4q^2$. This means that p is a multiple of four, so $p = 4n$ for some natural number n.

Since $4q^2 = p^2$ and $p = 4n$, this means that $4q^2 = (4n)^2 = 16n^2$, so $q^2 = 4n^2$. This means that q is a multiple of four as well. But since both p and q are multiples of four, this means that p and q share a common divisor other than 1 and -1, contradicting our initial assumption. We have reached a contradiction, so our assumption must have been incorrect. Thus $\sqrt{4}$ is irrational. ■

This proof has to be wrong, because $\sqrt{4} = 2$, which is indeed rational! Specifically, this proof contains two invalid steps that let it claim that $\sqrt{4}$ is irrational. What are the two invalid steps? Why doesn't this error occur in the similar proofs that $\sqrt{2}$ and $\sqrt{3}$ are irrational?

Problem Three: The Epimenides Paradox (12 points)

In logic, we assume that every statement is either true or false. However, some statements called *logical paradoxes* break this rule and can be neither true or false. For example, the statement "this statement is false" is a paradox – if it were true, it would have to be false, and if it were false, it would have to be true. The statement is therefore a paradox – it must be either true or false, but it can be neither true nor false.

One of the earliest paradoxes is the called the *Epimenides Paradox*, which is stated as follows:

Epimenides, a Cretan, says "All Cretans always lie."

According to the ancient Greeks, this statement is a paradox because Epimenides can neither tell the truth nor lie. A sketch of the argument is as follows:

> "If Epimenides tells the truth, then all Cretans always lie. Since Epimenides is himself a Cretan, then he must be lying, which is impossible because we know that Epimenides is telling the truth. Thus it is not possible for Epimenides to be telling the truth.
>
> If, on the other hand, Epimenides is lying, then his statement is false and all Cretans never lie. Since Epimenides himself is a Cretan, then he must be telling the truth, which is impossible because we know that he was lying. Thus it is not possible for Epimenides to be lying.
>
> Thus Epimenides must be neither lying nor telling the truth – a paradox!"

However, there is a flaw in the above line of reasoning, and despite its name the Epimenides Paradox is **not** a paradox.

Identify the flaw in this reasoning. Since this is not really a paradox, Epimenides must either be lying or telling the truth. Is Epimenides lying or telling the truth? If he's telling the truth, why doesn't his statement contradict itself? If he's lying, why doesn't his statement contradict itself?

Problem Four: Properties of Sets (20 points)

Below are four claims about sets. For each statement, if it is always true, prove it. If it is always false, prove that it is always false. If it is sometimes true and sometimes false, provide an example for which it is true and an example for which it is false and explain why your examples are correct.

To prove that two sets are equal, remember that you need to show that any element of the first set must also be an element of the second set and vice versa. Recall that this is equivalent to showing that the two sets are subsets of one another. It is **not** sufficient to use Venn diagrams or any other informal reasoning here. You need to formally prove each result.

i. If $A \in B$ and $B \in C$, then $A \in C$.

ii. If $\wp(A) = \wp(B)$, then $A = B$.

iii. $(A - B) \cup B = A$.

iv. $A \cap (B - A) \neq \emptyset$.

Problem Five: Ascending Sequences (12 points)

Suppose that you have an infinite sequence of real numbers x_0, x_1, …, x_n, … such that for any natural number i, $x_i < x_{i+1}$. Such a sequence is called an *ascending sequence*. For example, the series of natural numbers 0, 1, 2, 3, 4, … is such a sequence, as is the series 1, 2, 4, 8, 16, 32, … of powers of two.

Suppose that you have some number z that is sandwiched in-between two of the terms in the series; that is, there is some j such that $x_j < z < x_{j+1}$. Prove that z does not appear anywhere in the series by showing that there is no i such that $x_i = z$.

Problem Six: Modular Arithmetic (28 points)

Many programming languages support a modulus operator (in many languages, using the `%` operator), which gives the remainder when one number is divided by another. For example, 5 `%` 3 = 2, since three divides five with remainder two. Similarly, 17 `%` 6 = 5.

Multiple numbers might yield the same remainder when divided by some number. For example, 2, 5, 8, 11, 14, 17, etc. all leave a remainder of two when divided by three, and 1, 12, 23, 34, etc. all leave a remainder of one when divided by eleven. We can formalize the relationships between these numbers as follows. For any integer k, define the relation $\equiv_k$ as follows:

$$a \equiv_k b \text{ iff there exists an integer } q \text{ such that } a - b = kq$$

For example, $7 \equiv_3 4$, because $7 - 4 = 3 = 3 \cdot 1$, and $13 \equiv_4 5$ because $13 - 5 = 8 = 4 \cdot 2$. If $x \equiv_k y$, we say that *x is congruent to y modulo k*, hence the terminology in the checkpoint problem. In this problem, you will prove several properties of modular congruence.

i. Prove that for any integer x and any integer k, $x \equiv_k x$.

ii. Prove that for any integers x and y and any integer k, that if $x \equiv_k y$, then $y \equiv_k x$.

iii. Prove that for any integers x, y, and z and any integer k, that if $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.

The three properties you have just proven show that modular congruence is an *equivalence relation*. Equivalence relations are important throughout mathematics, and we'll see more examples of them later in the quarter.

Modular congruence plays well with arithmetic:

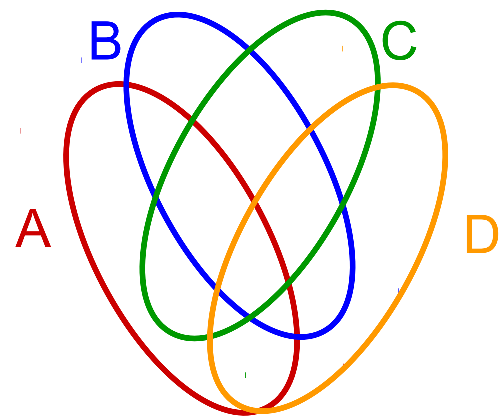
iv. Prove that for any integers w, x, y, z, and k, that if $x \equiv_k w$ and $y \equiv_k z$, then $x + y \equiv_k w + z$.

v. Prove that for any integers w, x, y, z, and k, that if $x \equiv_k w$ and $y \equiv_k z$, then $xy \equiv_k wz$.

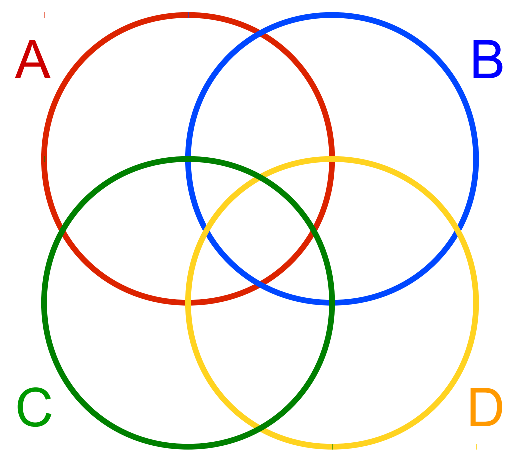
These last two results are important for how computers do arithmetic. Computers can't actually store arbitrary integers, because computers are inherently finite. Instead, when storing integers, computers typically represent them modulo some large power of two, such as 2^{32} or 2^{64}. The result that you have just proven shows that if the computer adds or multiplies numbers, the result will at least be correct modulo the large power of two, even if the actual result is too large to hold in memory.

Problem Seven: Venn Diagrams (12 Points)

In our first lecture, we saw the following picture, which represents a Venn diagram for four sets:



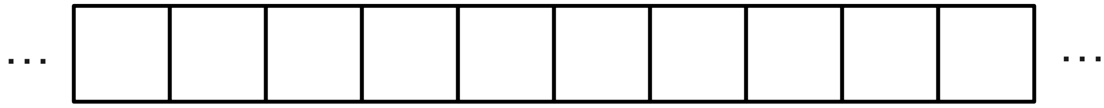
This picture is probably not what you would have initially expected. It might seem more reasonable to draw the Venn diagram this way:



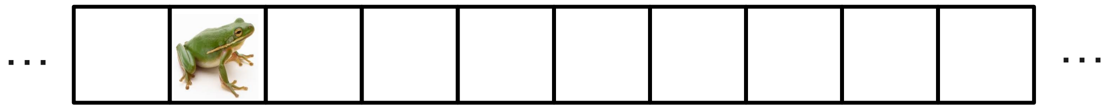
However, the way that these circles overlap is not sufficient to show all possible ways that four different sets can overlap. Come up with four sets *A*, *B*, *C*, and *D* such that there is no way to accurately represent the overlap of those four sets with the second Venn diagram, and explain why your sets have this property.

Problem Eight: The Quantum Frog (20 Points)

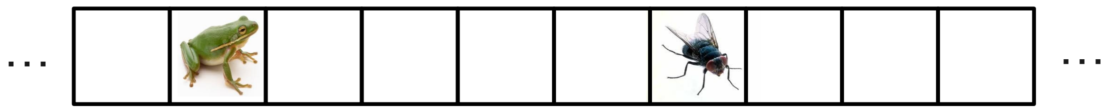
Imagine an infinitely long sequence of squares, such as below:



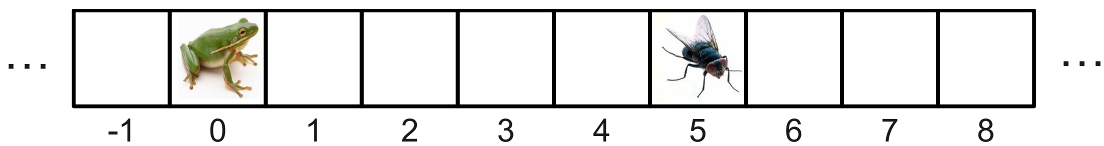
One of these squares contains a frog:



And another square contains a fly:



For simplicity, let's number all of the (infinitely many) squares by assigning each an integer. We'll say that the frog starts in position 0, and will assign positive integers to the squares to the right of the frog and negative numbers to the squares to the left of the frog. For example:



Now this frog is a very special kind of frog called a *quantum frog*. The quantum frog can hop across the squares forward and backward, but can only make jumps of two different lengths: 3 and 7. For example, to get to square five to eat the fly, the frog might might jump forward seven squares to square 7, forward seven squares again to square 14, then back three squares three times to squares 11, 8, and (finally) 5.

i. Prove that, starting at position 0, the quantum frog can move to any other square using only jumps of length 3 and 7.

Suppose that this quantum frog is very concerned about catching the fly as early as possible (this is a hungry quantum frog!) and wants to minimize the number of jumps he has to make to get from his starting point to the fly. All of the quantum frog's jumps take the same amount of time (this is, after all, a quantum frog!), so all the frog cares about is the total number of jumps made.

ii. Prove that in an optimal series of jumps from square 0 to square k, all jumps of the same distance must be made in the same direction. That is, all of the frog's jumps of distance 3 must be in the same direction and all of the frog's jumps of distance 7 must be in the same direction, but these two directions don't have to be the same.

iii. Prove that in an optimal series of jumps from square 0 to square k, the frog can never use jumps of size three more than six times.

Your results from (i), (ii), and (iii) can be used to devise a very efficient algorithm for finding the shortest number of hops required. Since you know that there can't be more than six jumps of size three and that all those jumps have to go in the same direction, you can just check, for each possible set of up to six jumps in each direction, how many remaining seven-hop jumps would be necessary. The minimum over all these options is the shortest sequence of jumps.

Problem Nine: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

i. How hard did you find this problem set? How long did it take you to finish?

ii. Does that seem unreasonably difficult or time-consuming for a five-unit class?

iii. Did you attend Monday's problem session? If so, did you find it useful?

iv. Did you read the course notes? If so, did you find them useful? Is there anything you would suggest that we could do to improve them?

v. How is the pace of this course so far? Too slow? Too fast? Just right?

vi. Is there anything in particular we could do better? Is there anything in particular that you think we're doing well?

Submission instructions

There are three ways to submit this assignment:

1. Hand in a physical copy of your answers at the start of class. This is probably the easiest way to submit if you are on campus.
2. Submit a physical copy of your answers in the filing cabinet in the open space near the handout hangout in the Gates building. If you haven't been there before, it's right inside the entrance labeled "Stanford Engineering Venture Fund Laboratories." There will be a clearly-labeled filing cabinet where you can submit your solutions.
3. Send an email with an electronic copy of your answers to the submission mailing list (cs103-spr1112-submissions@lists.stanford.edu) with the string "[PS1]" somewhere in the subject line.

If you are an SCPD student, we would strongly prefer that you submit solutions via email, especially for the checkpoint problems, so that we can get your solution graded and returned as quickly as possible. Please contact us if this will be a problem.

Extra Credit Problem: Ohnoez teh typoz! (2 Points)

Find an error in the course notes. Give the page number on which the error occurs, along with enough context for the course staff to fix the error.

If prior to this problem set going out you already found an error and emailed us, just make a note of that here and we'll give you the extra credit. You don't need to find another error.